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**Abstract**

In this note, we study the consistency of the Finite Volume kinetic scheme with reflections which is well-known to preserve the main physical properties of hyperbolic systems with source term. In particular, we prove that the numerical fluxes are consistent with the exact flux and are asymptotically consistent with the source term.

**Keywords**: kinetic scheme, finite volume, shallow water equations, consistency, USI method
**AMS Subject classification** : 65M08, 65M75, 76B07, 76M12, 76M28, 76N15

## 1 Introduction

During these last years, a great amount of works was devoted to the Finite Volume numerical approximation of hyperbolic equations with source terms, in particular for the Saint-Venant equations in a presence of a topography source term:

$$\partial_t U + \partial_x F(U) + B(U)\partial_x Z = 0 \tag{1}$$

where $U(t,x) \in \mathbb{R}^m$ stands for the unknown vector state, $F(U) \in \mathbb{R}^m$ is the flux function and $B(U)\partial_x Z$ is the source term.

Solving Equations (1) in presence of source term $B(U)\partial_x Z$ is a challenging problem since one has to provide a consistent discretization in order to preserve numerically some of or all the main physical features (stability properties) of the system. For instance, these properties are: non-negativity of the water height, preservation of steady states and the energy (entropy) inequalities for the Saint-Venant equations. One can found several works on this issue, see for instance [15, 17, 13, 16, 12, 19, 12, 3, 1, 5, 9, 6]), based on the upwind of sources at the interface which is quite flexible. However, none of these methods, except the kinetic scheme with reflections [19], are known to satisfy all of the stability properties.

The method is constructed from a (non physical) kinetic interpretation of the system leading to a Vlasov type kinetic equation including the source term as a potential. This kinetic equation, based on a well-suited density function, is then discretized where the source terms does not appear explicitly. As a matter of fact, the source terms appears in the microscopic interface fluxes which are constructed using an energetic balance. The so-called kinetic scheme with reflections is then obtained by taking the first, second and third moment (mass, momentum and energy) of the discrete kinetic equation as in the usual meaning of the hydrodynamic limit. Whenever the source term is of the form $B(U)Z'(x)$ (where $Z'(x)$ stands for $\dfrac{d}{dx}Z(x)$) as in Equation (1), the kinetic scheme with reflections preserve the physical properties of the system. Moreover, it has the remarkable property to treat naturally vacuum states (corresponding to drying and flooding flows for the Saint-Venant equations) and to remain stable when the numerical solution is close to those state.

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Nonetheless, whenever the source term is not under the "conservative form" $Z'(x)$, we cannot analytically obtain a well-suited density function to recover the exact preservation of still steady states and the discrete entropy inequalities (see for instance, [9, 6, 8]). However, one can prove that the domain invariant property is still satisfied while the equilibrium states are approximately obtained. In this case, even if we do not know how to prove that the scheme is entropy satisfying, numerical comparisons with exact solutions and experimental data show a perfect agreement as done by Bourdarias *et al* [8].

For both cases, the main difficult question concerns the consistency of the numerical fluxes with the exact fluxes and the asymptotic consistency with the source term (as defined by Bouchut [4]) that we prove in this paper. This results lead to the consistency of the kinetic scheme with reflections with System (1). The paper is organized as follows. In Section 2, for the sake of clarity and completeness, we recall the full construction of the kinetic scheme for the Saint-Venant Equations. In Section 3, we establish and prove the main result of this paper. Finally, we end with Section 4. Interested reader can also find in Appendix A, the full construction of the microscopic fluxes of the kinetic scheme.

## 2 The kinetic scheme for the Saint-Venant equations

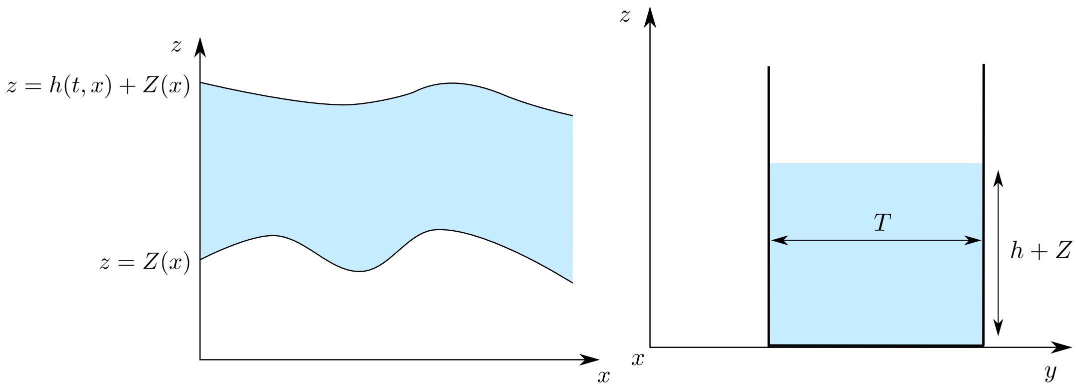
Besides the fact that the shallow water equations are well-suited for the simulation of free surface flows (and widely used for engineering applications), it is used as a generic example of hyperbolic system for which one can easily add source terms to deal with storm sewers, waste, supply pipes in hydroelectric installations, sedimentation, rivers, coastal domains, oceans as well as avalanches problems.

These equations were originally written by A.J.C. de Saint-Venant in [2] and studied by several authors. For instance, Gerbeau and Perthame [14] study the full derivation of the one dimensional viscous and inviscid shallow water equations from the 2D Navier-Stokes equations including a small friction term on a flat bottom. In the same spirit, Marche [18] propose a new two dimensional viscous and non viscous shallow water equations in a rotating framework with varying topography including Coriolis force, friction (linear and quadratic), and capillary effects. Recently, Ersoy [10, 11] study the derivation of a new 1D shallow water like equations for incompressible and compressible pipe flows.

The one-dimensional Saint-Venant equations with a topography source term $Z(x)$ is well-adapted for rectangular rivers or channel. The hyperbolic system:

$$\left\{\begin{array}{lcl} \partial_t h + \partial_x q & = & 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} + g\dfrac{h^2}{2}\right) & = & -gh\partial_x Z, \end{array}\right. \tag{2}$$

describes the unidirectional flow through the water height $h(t,x) \geqslant 0$ and its discharge $q(t,x) = \dfrac{h(t,x)}{u(t,x)} \in \mathbb{R}$ at time $t \geqslant 0$ and at point $x \in \mathbb{R}$ where $u(t,x)$ stands for the velocity. Here $g$ is the gravity constant and $Z(x)$ is the topography elevation from the surface $z = 0$ (see figure 1). In the sequel, we note $\mathbf{U} = \begin{pmatrix} h \\ q \end{pmatrix}$ the unknown vector state, $F(\mathbf{U}) = \begin{pmatrix} q \\ \dfrac{q^2}{h} + g\dfrac{h^2}{2} \end{pmatrix}$ the flux function and $B(\mathbf{U}) = \begin{pmatrix} 0 \\ -gh \end{pmatrix} \partial_x Z$ the source term.




(a) Configuration (b) z-plane

Figure 1: Geometric characteristics

The Saint-Venant system has the following physical and mathematical properties:

**Theorem 2.1.**

1. *System (2) is strictly hyperbolic on the set* $\{h(t, x) > 0\}$ .
2. *For smooth solutions, the velocity* $u = q/h$ *satisfies:*
$$\partial_t u + \partial_x \left( \frac{u^2}{2} + gh + gZ \right) = 0$$
*where the quantity* $\frac{u^2}{2} + gh + gZ$ *is called the total head.*
3. *The still water steady state, for* $u = 0$*, reads:*
$$h + Z = h_0 \ , \tag{3}$$
*for some constant* $h_0 > 0$.
4. *System (2) admits a mathematical entropy:*
$$\mathcal{E}(h, q) = \frac{q^2}{2h} + g\frac{h^2}{2} + gZh \ ,$$
*which satisfies the entropy inequality*
$$\partial_t \mathcal{E} + \partial_x \left( (\mathcal{E} + g\frac{h^2}{2})u \right) \leqslant 0. \tag{4}$$

## 2.1 The mathematical kinetic interpretation

Starting with a given real function $\chi : \mathbb{R} \to \mathbb{R}$ satisfying the properties:

$$\chi(\omega) = \chi(-\omega) \geqslant 0, \quad \int_{\mathbb{R}} \chi(\omega) d\omega = 1, \quad \int_{\mathbb{R}} \omega^2 \chi(\omega) d\omega = \frac{g}{2} \ , \tag{5}$$

we define the density of particles

$$\mathcal{M}(t, x, \xi) = \sqrt{h(t, x)} \chi \left( \frac{\xi - u(t, x)}{\sqrt{h(t, x)}} \right)$$

which allows to provide a (non physical) kinetic interpretation of System (2) through the *macro-micro*scopic kinetic relations:

$$h = \int_{\mathbb{R}} \mathcal{M}(t,x,\xi)\,d\xi\,, \quad q = \int_{\mathbb{R}} \xi\mathcal{M}(t,x,\xi)\,d\xi\,, \quad \frac{q^2}{h} + g\frac{h^2}{2} = \int_{\mathbb{R}} \xi^2\mathcal{M}(t,x,\xi)\,d\xi\ . \tag{6}$$

Thus, one has

**Theorem 2.2** (Kinetic interpretation)**.** *$(h,q)$ is a strong solution of System (2) if and only if $\mathcal{M}$ satisfies the kinetic equation:*

$$\partial_t\mathcal{M} + \xi\cdot\partial_x\mathcal{M} - g\partial_x Z\,\partial_\xi\mathcal{M} = \mathcal{K}(t,x,\xi), \tag{7}$$

*for a collision term $\mathcal{K}(t,x,\xi)$ which satisfies for $(t,x)$ a.e.*

$$\int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} \mathcal{K}(t,x,\xi)\,d\xi = 0\,,$$

**Remark 2.1.** As done by Ersoy *et al* [9, 6, 8], one can easily adapt the kinetic interpretation for a general source term

$$\varphi = \partial_x Z + f\partial_x g + F\ .$$

## 2.2 Construction of the kinetic scheme

Let us consider the following mesh on $\mathbb{R}$. Cells are denoted for every $i \in \mathbb{Z}$, by $m_i = (x_{i-1/2}, x_{i+1/2})$, with $x_i = \dfrac{x_{i-1/2} + x_{i+1/2}}{2}$ and $\Delta x_i = x_{i+1/2} - x_{i-1/2}$ the space step (see figure 2).

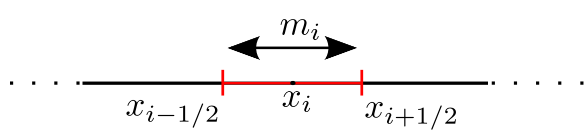



Figure 2: The space discretisation.

We also consider a time discretization $t^n$ defined by $t^{n+1} = t^n + \Delta t^n$ with $\Delta t^n$ the time step.

We denote $\mathbf{U}_i^n = (h_i^n, q_i^n)$, $u_i^n = \dfrac{q_i^n}{h_i^n}$, $\mathcal{M}_i^n$ the cell-centered approximation of $\mathbf{U} = (h,q)$, $u$ and $\mathcal{M}$ on the cell $m_i$ at time $t^n$.

On a time interval $[t^n, t^{n+1}]$ and on the cell $m_i$, the kinetic equation (7) writes:

$$\left\{\begin{array}{ll} \partial_t\mathcal{M} + \xi\cdot\partial_x\mathcal{M} - g\partial_x Z\,\partial_\xi\mathcal{M} = \mathcal{K}(t,x,\xi) & \text{for } x \in m_i\,,\ t \in (t^n, t^{n+1})\,,\ \xi \in \mathbb{R}, \\ \mathcal{M}(t^n,x,\xi) = \mathcal{M}_i^n(\xi) & \text{for } x \in m_i\,,\ \xi \in \mathbb{R}\ . \end{array}\right. \tag{8}$$

We define the piecewise constant representation of $Z$

$$Z(x) = Z_i \mathbb{1}_{m_i}(x) \tag{9}$$

where $Z_i = \dfrac{1}{\Delta x_i}\displaystyle\int_{m_i} Z(x)\,dx$ for instance.

Neglecting the collision kernel, the kinetic equation (8) simply reads:

$$\left\{\begin{array}{ll} \dfrac{\partial}{\partial t}f + \xi\cdot\dfrac{\partial}{\partial x}f = 0 & \text{for } x \in m_i\,,\ t \in (t^n, t^{n+1})\,,\ \xi \in \mathbb{R}, \\ f(t^n,x,\xi) = \mathcal{M}_i^n(\xi) & \text{for } x \in m_i\,,\ \xi \in \mathbb{R}\ . \end{array}\right. \tag{10}$$

This equation is a linear transport equation whose explicit discretisation may be done directly by the following way. A finite volume discretisation of Equation (10) leads to:

$$\forall \xi \in \mathbb{R}\,,\, \forall x \in m_i\,,\; f(t^{n+1},x,\xi) = f_i^{n+1}(\xi) = \mathcal{M}_i^n(\xi) + \frac{\Delta t}{\Delta x_i}\,\xi\,\left(\mathcal{M}^-_{i+\frac{1}{2}}(\xi) - \mathcal{M}^+_{i-\frac{1}{2}}(\xi)\right)\;. \tag{11}$$

The microscopic fluxes are defined (see Appendix A) by

$$\begin{aligned}
\mathcal{M}^-_{i+1/2}(\xi) &= \mathbb{1}_{\{\xi>0\}}\mathcal{M}_i^n(\xi) + \mathbb{1}_{\left\{\xi<0,|\xi|^2-2g\Delta Z_{i+1/2}<0\right\}}\mathcal{M}_i^n(-\xi)\\
&+ \mathbb{1}_{\left\{\xi<0,|\xi|^2-2g\Delta Z_{i+1/2}>0\right\}}\mathcal{M}_{i+1}^n\left(-\sqrt{|\xi|^2-2g\Delta Z_{i+1/2}}\right)\\
\\
\mathcal{M}^+_{i+1/2}(\xi) &= \mathbb{1}_{\{\xi<0\}}\mathcal{M}_{i+1}^n(\xi) + \mathbb{1}_{\left\{\xi>0,|\xi|^2+2g\Delta Z_{i+1/2}<0\right\}}\mathcal{M}_{i+1}^n(-\xi)\\
&+ \mathbb{1}_{\left\{\xi>0,|\xi|^2+2g\Delta Z_{i+1/2}>0\right\}}\mathcal{M}_i^n\left(|\xi|^2+2g\Delta Z_{i+1/2}\right)
\end{aligned} \tag{12}$$

and take into account the discontinuity of the source term $Z$ at the cell interface $x_{i+1/2}$ through the term

$$\Delta Z_{i+1/2} = Z_{i+1} - Z_i\;.$$

This is the principle of interfacial source upwind.

Keeping in mind Identities (6), we integrate the discretised kinetic equation (11) against 1 and $\xi$ to obtain the kinetic scheme:

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \frac{\Delta t^n}{h_i}\left(\boldsymbol{F}^-_{i+\frac{1}{2}} - \boldsymbol{F}^+_{i-\frac{1}{2}}\right)\;. \tag{13}$$

where the numerical fluxes are thus defined by the kinetic fluxes as follows:

$$\boldsymbol{F}^\pm_{i+\frac{1}{2}} \stackrel{def}{=} \int_{\mathbb{R}} \xi \begin{pmatrix} 1 \\ \xi \end{pmatrix} \mathcal{M}^\pm_{i+\frac{1}{2}}(\xi)\,d\xi\;. \tag{14}$$

**Remark 2.2.** Let us emphasize that, one has an explicit dependency on the jump of $Z$ through the interface $x_{i+1/2}$. Thus, we should write fluxes as follows:

$$\boldsymbol{F}^\pm_{i+\frac{1}{2}}(\mathbf{U}_i,\mathbf{U}_{i+1}) = \boldsymbol{F}^\pm_{i+\frac{1}{2}}(\mathbf{U}_i,\mathbf{U}_{i+1},\Delta Z_{i+1/2}) \quad \text{with} \quad \mathcal{M}^\pm_{i+1/2}(\xi) = \mathcal{M}^\pm_{i+1/2}(\xi;\mathbf{U}_i,\mathbf{U}_{i+1},\Delta Z_{i+1/2})$$

At this stage of the construction of the scheme, the choice of the function $\chi$ is crucial to satisfy some physical (stability) properties of Equations (2). There exists essentially two type of $\chi$-function. The first one

$$\chi(w) = \frac{\sqrt{2}}{\pi\sqrt{g}}\left(1 - \frac{w^2}{2g}\right)_+^{1/2}\;. \tag{15}$$

is motivated by the fact to construct a numerical scheme which preserves non-negativity of the water height, still water steady states (see Equation (3)) and in-cell entropy inequality (see Equation (4)). The second one

$$\chi(w) = \frac{1}{2\sqrt{3}}\mathbb{1}_{[-\sqrt{3},\sqrt{3}]}(w) \tag{16}$$

is motivated to compute explicitly and easily the numerical fluxes (14). For more details on this issue, we refer to [19] and [9, 8].

However, for a general compactly supported $\chi$-function, one has

**Proposition 2.1.** *Let $\chi$ be a compactly supported function verifying (5) and denote $[-M,M]$ its support. The kinetic scheme (13)-(14) has the following properties:*

1. *The kinetic scheme is $h$-conservative scheme,*

2. *Assume the following CFL condition*

$$\Delta t^n \max_i \left(|u_i^n| + M\sqrt{h_i^n}\right) \leqslant \max_i \Delta x_i$$

*holds. Then the kinetic scheme keeps the water height $h$ positive i.e:*

$$\textit{if, for every } i \in \mathbb{Z}\,,\ h_i^0 \geqslant 0 \textit{ then, for every } i \in \mathbb{Z}\,,\ h_i^n \geqslant 0.$$

3. *The kinetic scheme treats "naturally" flooding zones.*

*Proof.* We refer to [9, 8] for the details of the proof. □

# 3 Main result

In this section, we present the main result of this paper. It concerns the consistency with the exact flux and the asymptotic consistency of the source term. In particular, it means that the kinetic scheme (13) with (14)-(12) is consistent with Equation (2). We assume in the rest of the paper that $\delta x = \Delta x_i$.

Following [4] and Remark 2.2, we set

$$\begin{aligned}
F_l(\mathbf{U}_i, \mathbf{U}_{i+1}, \Delta Z_{i+1/2}) &= F^-_{i+1/2}(\mathbf{U}_i, \mathbf{U}_{i+1}, \Delta Z_{i+1/2}) = \int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} \mathcal{M}^-_{i+1/2}(\xi; \mathbf{U}_i, \mathbf{U}_{i+1}, \Delta Z_{i+1/2})\, d\xi\ , \\
F_r(\mathbf{U}_i, \mathbf{U}_{i+1}, \Delta Z_{i+1/2}) &= F^+_{i+1/2}(\mathbf{U}_i, \mathbf{U}_{i+1}, \Delta Z_{i+1/2}) = \int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} \mathcal{M}^+_{i+1/2}(\xi; \mathbf{U}_i, \mathbf{U}_{i+1}, \Delta Z_{i+1/2})\, d\xi\ .
\end{aligned}$$

Let us consider the $\chi$-function given by (15) and let us note $\Delta Z_{l,r} = Z_r - Z_l$ for any given $(Z_l, Z_r) \in \mathbb{R}^2$.

Then,

**Theorem 3.1.** *The first order three-point kinetic scheme (13)-(14) with the $\chi$-function (15) is consistent with Equations (2), i.e. the numerical fluxes satisfy the consistency with the exact flux*

$$\boldsymbol{F}_r(\boldsymbol{U}, \boldsymbol{U}, 0) = \boldsymbol{F}_l(\boldsymbol{U}, \boldsymbol{U}, 0) = F(\boldsymbol{U}) \quad \textit{for any } \boldsymbol{U} \in \mathbb{R}^2$$

*and the asymptotic consistency with the source term*

$$\boldsymbol{F}_r(\boldsymbol{U}_l, \boldsymbol{U}_r, \Delta Z_{l,r}) - \boldsymbol{F}_l(\boldsymbol{U}_l, \boldsymbol{U}_r, \Delta Z_{l,r}) = -B(\boldsymbol{U})\Delta Z_{l,r} + O(\Delta Z^p_{l,r})\ , p \geqslant \frac{3}{2} \tag{17}$$

*as*

$$(\boldsymbol{U}_l, \boldsymbol{U}_r, \Delta Z_{l,r}) \to (\boldsymbol{U}, \boldsymbol{U}, 0)\ .$$

Before to prove this consistency theorem, let us make the following useful remarks:

**Remark 3.1.** Noting $(\Delta Z_{i+1/2})_+$ and $(\Delta Z_{i+1/2})_-$ the positive and negative part of the real number $\Delta Z_{i+1/2}$, one has

- $\xi < 0$ and $|\xi|^2 - 2g\Delta Z_{i+1/2} < 0$ if and only if $\xi \in \left(-\sqrt{2g(\Delta Z_{i+1/2})_+}, 0\right)$,
- $\xi < 0$ and $|\xi|^2 - 2g\Delta Z_{i+1/2} > 0$ if and only if $\xi < -\sqrt{2g(\Delta Z_{i+1/2})_+}$,
- $\xi > 0$ and $|\xi|^2 + 2g\Delta Z_{i+1/2} < 0$ if and only if $\xi \in \left(0, \sqrt{2g(\Delta Z_{i+1/2})_-}\right)$,
- $\xi > 0$ and $|\xi|^2 + 2g\Delta Z_{i+1/2} > 0$ if and only if $\xi > \sqrt{2g(\Delta Z_{i+1/2})_-}$.

Thus, one can write fluxes (12) as

$$\begin{aligned}
\mathcal{M}^-_{i+1/2}(\xi) &= \mathbb{1}_{\{\xi>0\}}(\xi)\mathcal{M}^n_i(\xi) + \mathbb{1}_{\left\{-\sqrt{2g(\Delta Z_{i+1/2})_+}<\xi<0\right\}}(\xi)\mathcal{M}^n_i(-\xi) \\
&+ \mathbb{1}_{\left\{\xi<-\sqrt{2g(\Delta Z_{i+1/2})_+}\right\}}(\xi)\mathcal{M}^n_{i+1}\left(-\sqrt{|\xi|^2-2g\Delta Z_{i+1/2}}\right) \\
\mathcal{M}^+_{i+1/2}(\xi) &= \mathbb{1}_{\{\xi<0\}}(\xi)\mathcal{M}^n_{i+1}(\xi) + \mathbb{1}_{\left\{0<\xi<\sqrt{2g(\Delta Z_{i+1/2})_-}\right\}}(\xi)\mathcal{M}^n_{i+1}(-\xi) \\
&+ \mathbb{1}_{\left\{\xi>\sqrt{2g(\Delta Z_{i+1/2})_-}\right\}}(\xi)\mathcal{M}^n_i\left(\sqrt{|\xi|^2+2g\Delta Z_{i+1/2}}\right)
\end{aligned} \tag{18}$$

**Remark 3.2.** Let $x \in (x_i, x_{i+1/2}), \xi$ such that $|\xi|^2 + 2g(Z(x) - Z(x_i)) > 0$.
Assuming $Z_i = Z(x_i) + O(\delta x)$, one has

$$\mathcal{M}(x+\delta x, \varphi(x+\delta x, \xi)) = \mathcal{M}(x,\xi) + \partial_x\mathcal{M}(x,\xi)\delta x + \frac{gZ'(x)}{\xi}\partial_\xi\mathcal{M}(x,\xi)\delta x + \partial^2_{xx}\mathcal{M}\frac{\delta x^2}{2} + O(\delta x^3) \tag{19}$$

where

$$\varphi(x,\xi) = \sqrt{|\xi|^2 + 2g(Z(x) - Z(x_i))}\ .$$

Indeed, setting

$$\varphi(x,\xi) = \sqrt{|\xi|^2 + 2gZ'(x_i)(x-x_i) + O((x-x_i)^2)}$$

we deduce

$$\partial_x\varphi(x,\xi) = \frac{gZ'(x_i) + O(x-x_i)}{\varphi(x,\xi)} \quad \text{and} \quad \partial_x\varphi(x_i,\xi) = \frac{gZ'(x_i)}{\xi},\ \xi \neq 0\ .$$

Moreover, expanding $\mathcal{M}$ with respect to $x$, we get:

$$\mathcal{M}(x+\delta x, \varphi(x+\delta x,\xi)) = \mathcal{M}(x,\xi) + \partial_x\mathcal{M}(x,\varphi(x,\xi))\delta x + \partial^2_{xx}\mathcal{M}\frac{\delta x^2}{2} + O(\delta x^3)$$

where

$$\partial_x\mathcal{M}(x,\varphi(x,\xi)) = (\partial_x\mathcal{M})(x,\xi) + \partial_x\varphi(x,\xi)(\partial_\xi\mathcal{M})(x,\xi)\ .$$

Gathering these simple computations, we obtain the expression (19).

*Proof.* Let us set $\boldsymbol{W} = (\mathbf{U}_l, \mathbf{U}_r, \Delta Z_{l,r})$ and assume $\Delta Z_{l,r} = 0$. Then, the identity

$$\boldsymbol{F}_r(\boldsymbol{W}) = \boldsymbol{F}_l(\boldsymbol{W}) = F(\mathbf{U})$$

holds for any $\mathbf{U} \in \mathbb{R}^2$ since the kinetic scheme is conservative in that case. It suffices to replace in the expression of the numerical fluxes (14) $\Delta Z_{l,r}$ by 0.

Now, let us assume $\Delta Z_{l,r} \neq 0$. The kinetic scheme being $h$-conservative (see Proposition 2.1 1.), the first component of the relation (17) always holds. Indeed, an easy computation, using the change of variable $\mu = |\xi|^2 - 2g(Z_r - Z_l)$, allows us to show that:

$$\boldsymbol{F}^h_r(\boldsymbol{W}) - \boldsymbol{F}^h_l(\boldsymbol{W}) = 0\ .$$

To show the asymptotic consistency with the source for the second component of Equation (17), we proceed at the microscopic scale using the fluxes (18) (see Remark 3.1). As done before, let us first set

$$\mathcal{M}_l(\xi, \mathbf{U}_i, \mathbf{U}_{i+1}, \Delta Z_{i+1/2}) = \mathcal{M}^+_{i+1/2}(\xi) := \mathcal{M}^+_{i+1/2}(\xi; \mathbf{U}_i, \mathbf{U}_{i+1}, \Delta Z_{i+1/2})\ ,$$

$$\mathcal{M}_r(\xi, \mathbf{U}_i, \mathbf{U}_{i+1}, \Delta Z_{i+1/2}) = \mathcal{M}^+_{i+1/2}(\xi) := \mathcal{M}^-_{i+1/2}(\xi; \mathbf{U}_i, \mathbf{U}_{i+1}, \Delta Z_{i+1/2})$$

and let us note the flux difference

$$\delta\mathcal{M}(\xi) = \mathcal{M}_r(\xi) - \mathcal{M}_l(\xi)$$

omitting the dependency on $\mathbf{U}$ and $\Delta Z$.

**Assume $\xi > 0$.**

Noting $\Delta Z = Z_r - Z_l$, the flux difference reduces to

$$\begin{aligned} \delta\mathcal{M}(\xi) &= \mathbb{1}_{\left\{0<\xi<\sqrt{2g(\Delta Z)_-}\right\}}(\xi)\left(\mathcal{M}_r^n(-\xi) - \mathcal{M}_l^n(\xi)\right) \\ &\quad + \mathbb{1}_{\left\{\xi>\sqrt{2g(\Delta Z)_-}\right\}}(\xi)\left(\mathcal{M}_l^n\left(\sqrt{|\xi|^2+2g\Delta Z}\right) - \mathcal{M}_l^n(\xi)\right) \end{aligned}$$

Keeping in mind Remark 3.2, one can expand the first and the second term with respect to $x$ in the above equation. We get

$$\mathcal{M}_r^n(-\xi) - \mathcal{M}_l^n(\xi) = \mathcal{M}_l^n(-\xi) - \mathcal{M}_l^n(\xi) + \partial_x\mathcal{M}_l(-\xi)\delta x + O(\delta x^2)$$

and

$$\begin{aligned} \mathcal{M}_l^n\left(\sqrt{|\xi|^2+2g\Delta Z}\right) - \mathcal{M}_l^n(\xi) &= \mathcal{M}(x_l, \varphi(x_l+\delta x, \xi)) - \mathcal{M}(x_l, \xi) \\ &= \frac{gZ'(x_l)}{\xi}\partial_\xi\mathcal{M}_l(\xi)\delta x + O(\delta x^2)\ . \end{aligned}$$

Thus, we deduce that

$$\begin{aligned} \delta\mathcal{M}(\xi) &= \mathbb{1}_{\left\{0<\xi<\sqrt{2g(\Delta Z)_-}\right\}}(\xi)\left(\mathcal{M}_l^n(-\xi) - \mathcal{M}_l^n(\xi) + \partial_x\mathcal{M}_l(-\xi)\delta x\right) \\ &\quad + \mathbb{1}_{\left\{\xi>\sqrt{2g(\Delta Z)_-}\right\}}(\xi)\left(\frac{gZ'(x_l)}{\xi}\partial_\xi\mathcal{M}_l(\xi)\delta x\right) + O(\delta x^2). \end{aligned} \tag{20}$$

**Assume $\xi < 0$.**

Proceeding as done before, we get

$$\begin{aligned} \delta\mathcal{M}(\xi) &= \mathbb{1}_{\left\{-\sqrt{2g(\Delta Z)_+}\right\}<\xi<0}(\xi)\left(\mathcal{M}_l^n(\xi) - \mathcal{M}_l^n(-\xi) + \partial_x\mathcal{M}_l(\xi)\delta x\right) \\ &\quad + \mathbb{1}_{\left\{\xi<-\sqrt{2g(\Delta Z)_+}\right\}}(\xi)\left(\frac{gZ'(x_l)}{\xi}\partial_\xi\mathcal{M}_l(\xi)\delta x\right) + O(\delta x^2). \end{aligned} \tag{21}$$

Gathering results (20) and (21), the difference $\delta M$ is, for any $\xi \in \mathbb{R}$ :

$$\begin{aligned} \delta\mathcal{M}(\xi) &= \mathbb{1}_{\left\{-\sqrt{2g(\Delta Z)_+}<\xi<\sqrt{2g(\Delta Z)_-}\right\}}(\xi)\left(\left(\mathcal{M}_l^n(s\xi) - \mathcal{M}_l^n(-s\xi)\right) + \partial_x\mathcal{M}_l(s\xi)\delta x\right) \\ &\quad + \mathbb{1}_{\left\{\xi>\sqrt{2g(\Delta Z)_-}\right\}}(\xi)\left(\frac{gZ'(x_l)}{\xi}\partial_\xi\mathcal{M}_l(\xi)\delta x\right) \\ &\quad + \mathbb{1}_{\left\{\xi<-\sqrt{2g(\Delta Z)_+}\right\}}(\xi)\left(\frac{gZ'(x_l)}{\xi}\partial_\xi\mathcal{M}_l(\xi)\delta x\right) + O(\delta x^2). \end{aligned} \tag{22}$$

where

$$s = \operatorname{sgn}(\Delta Z) = \begin{cases} 1 & \text{if} \quad \Delta Z > 0\ , \\ -1 & \text{if} \quad \Delta Z < 0\ . \end{cases}$$

In order to show the asymptotic consistency with the source term, it remains to compute the integral $\int_{\mathbb{R}} \xi^2 \delta\mathcal{M}(\xi)\ d\xi$ which is, by construction,

$$\int_{\mathbb{R}} \xi^2 \delta\mathcal{M}(\xi)\ d\xi = \boldsymbol{F}_r^q(\boldsymbol{W}) - \boldsymbol{F}_l^q(\boldsymbol{W})$$

where $\boldsymbol{F}^q$ stands for the second component of the flux of Equation (2).

Assuming $\Delta Z < 0$ (the case $\Delta Z > 0$ being similar, it will be not treated), the flux difference (22) is now

$$\begin{aligned} \delta\mathcal{M}(\xi) &= \left(\frac{gZ'(x_l)}{\xi}\partial_\xi\mathcal{M}_l(\xi)\delta x\right) - \mathbb{1}_{\left\{-\infty<\xi<\sqrt{2g(\Delta Z)_-}\right\}}(\xi)\left(\frac{gZ'(x_l)}{\xi}\partial_\xi\mathcal{M}_l(\xi)\delta x\right) \\ &\quad + \mathbb{1}_{\left\{0<\xi<\sqrt{2g(\Delta Z)_-}\right\}}(\xi)\left(\mathcal{M}_l^n(\xi) - \mathcal{M}_l^n(-\xi) + \partial_x\mathcal{M}_l(\xi)\delta x\right) + O(\delta x^2). \end{aligned} \tag{23}$$

Then, integrating Equation (23) against $\xi^2$, we get:

$$\boldsymbol{F}_r^q(\boldsymbol{W}) - \boldsymbol{F}_l^q(\boldsymbol{W}) := -gh_l Z'(x_l)\delta x + O(\delta x^p)$$

where $p \geqslant 3/2$. Indeed, choosing a lower and upper bound for the $\chi$-function (15), for instance,

$$\frac{1}{2}\mathbb{1}_{\left\{-\frac{1}{2}<\xi<\frac{1}{2}\right\}}(w) \leqslant \chi(w) \leqslant \mathbb{1}_{\{-1<\xi<1\}}(w) \ ,$$

an easy computation show that

$$\int_{\mathbb{R}} \xi^2 \left(-\mathbb{1}_{\{-\infty<\xi<\sqrt{2g(\Delta Z)_-}\}}(\xi)\left(\frac{gZ'(x_l)}{\xi}\partial_\xi \mathcal{M}_l(\xi)\delta x\right)\right)\ d\xi$$
$$+\int_{\mathbb{R}} \xi^2 \left(\mathbb{1}_{\{0<\xi<\sqrt{2g(\Delta Z)_-}\}}(\xi)\left(\mathcal{M}_l^n(\xi) - \mathcal{M}_l^n(-\xi) + \partial_x \mathcal{M}_l(\xi)\delta x\right)\right)\ d\xi \geqslant O(\delta x^{3/2})$$

which ends the proof. □

**Corollary 3.1.** *The first order three-point kinetic scheme* (13)-(14) *with the* $\chi$*-function* (16) *is also consistent with Equations* (2).

*Proof.* Integrating Equation (23) against $\xi^2$, all derivatives on $x$ or $\xi$ can be reported by integration by part or using the Leibniz theorem. As a consequence, the result still holds for any compactly supported $\chi$-function satisfying relations (5). □

Noting $\delta F := \dfrac{\boldsymbol{F}_r^q(\boldsymbol{W}) - \boldsymbol{F}_l^q(\boldsymbol{W})}{-gh_l Z'(x_l)}$, we look for $p$ at the numerical level to illustrate the fact that the estimate leading to $p \geqslant \dfrac{3}{2}$ is rather sharp in the proof. To this end, we compute

$$1 - \delta F = O(\delta x^{p-1}) \ .$$

We display on figure 3 the numerical asymptotic behavior of $\log(1-\delta F)$ as a function of $\log(\delta x)$. We found that at the numerical level the order of $1-\delta F$ is $p-1 \approx 1$, i.e. $p > \dfrac{3}{2}$ as predicted in the proof.

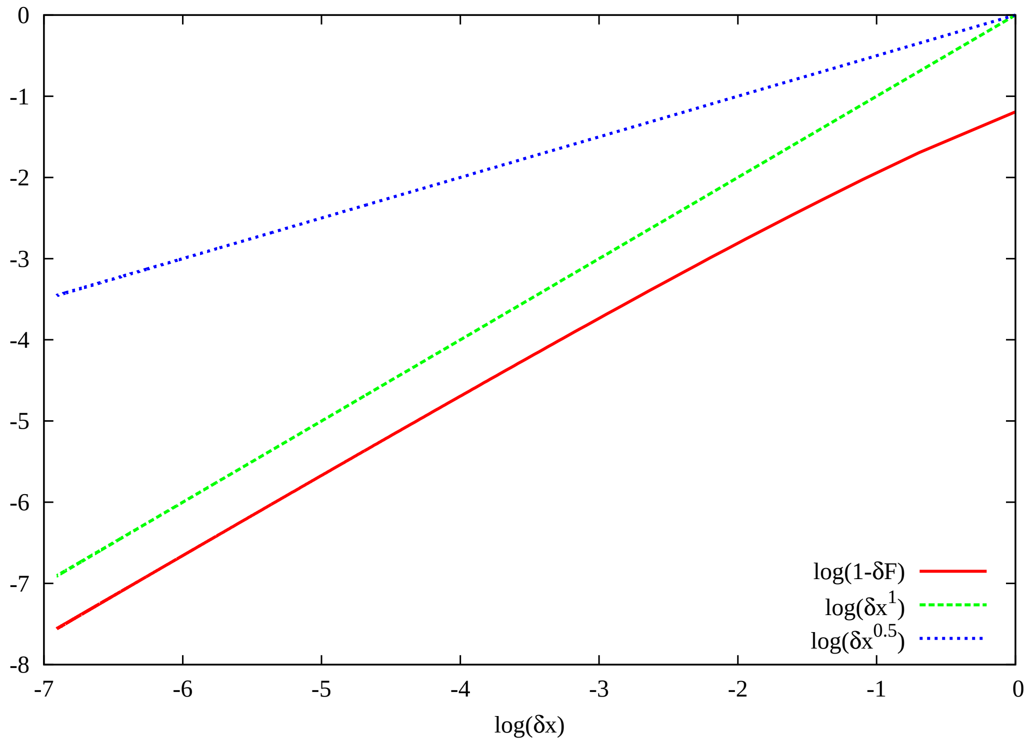



Figure 3: Numerical asymptotic consistency order using the $\chi$-function (16)

# 4 Concluding remark and further extensions

In this paper, we have shown that the kinetic scheme with reflections introduced by Perthame and Simeoni [19] is consistent with the exact fluxes and is asymptotically consistent with the source term. We have also establish that this consistency result is still true in the case of the indicator $\chi$-function (16). Moreover, let us also add that replacing $Z$ by a more general source term the consistency still holds. In particular, it justify the generalized kinetic scheme with reflections introduced by Ersoy *et al* [5, 6, 8] in the context of unsteady mixed flows in closed water pipes (see also [10, 11]) and in the framework of a two-layer approach for air entrainment in closed water pipes [7].

Besides the consistency property, the kinetic scheme using the $\chi$-function (15) satisfy all the stability properties of Equation (2): water-height remains non-negative, the energy inequality is satisfied and still water steady states are exactly preserved. Therefore, this scheme is well-adapted to the numerical integration of hyperbolic system with a conservative source term.

# A Computation of the numerical microscopic fluxes

For the sake of completeness, following [5, 9], we detail here the way to compute the microscopic fluxes $\mathcal{M}^{\pm}_{i+\frac{1}{2}}$ at the interface. It is justified through the characteristic of the scalar two-dimensional transport Equation (7) (neglecting the collision kernel). One can found a more general approach in [9, Chapter 2].

Let $s \in (t_n, t_{n+1})$ a time variable and $i$ the index of a cell $m_i$. The characteristic curves $\Xi(s)$ et $X(s)$ of

Equations (7) write:

$$\begin{aligned} \frac{d\,\Xi}{ds} &= -g\partial_x Z(X(s)) \ , && (24) \\ \frac{d\,X}{ds} &= \Xi(s) && (25) \end{aligned}$$

with the final data

$$\begin{cases} \Xi(t_{n+1}) &= \xi, \\ X(t_{n+1}) &= x_{i+1/2} \end{cases}$$

for some constant $\xi$.

Multiplying Equation (24) by $\Xi$, using Equation (25), we obtain the conservation of the kinetic enrgy $\frac{\Xi(s)^2}{2}$ and the potential energy $gZ(X(s))$

$$\frac{d}{ds}\left(\frac{\Xi(s)^2}{2} + gZ(X(s)\right) = 0 \ .$$

It yields to

$$\frac{|\xi_n|^2}{2} - \frac{|\xi_{n+1}|^2}{2} = g\Delta Z_{i+1/2} \tag{26}$$

where we make use of the notations

$$\xi_n = \Xi(t_n), \quad \xi_{n+1} = \Xi(t_{n+1}) \quad \text{and} \quad \Delta Z_{i+1/2} = Z(X(t_{n+1})) - Z(X(t_n))$$

which corresponds to the trajectory of a particle initially located at $X(t_n)$ with a velocity $\xi_n$ and arriving at a point $X(t_{n+1})$ with a velocity $\xi_{n+1}$.

Since $Z$ is assumed to be constant per cell (9), the trajectory are given by

$$\Xi(s) = \xi_{n+1} \quad \text{and} \quad X(s) = \xi_{n+1}(s - t_{n+1}) + x_{i+1/2} \tag{27}$$

with the data

$$\begin{cases} \Xi(t_{n+1}) &= \xi_{n+1} \ , \\ X(t_{n+1}) &= x_{i+1/2} \ . \end{cases}$$

Thus, trajectories are discontinuous in the $(X, \Xi)$-plane and composed of segments which are parallels to the $x$-axis. As a consequence, one has only three possible case as displayed on figure 4.

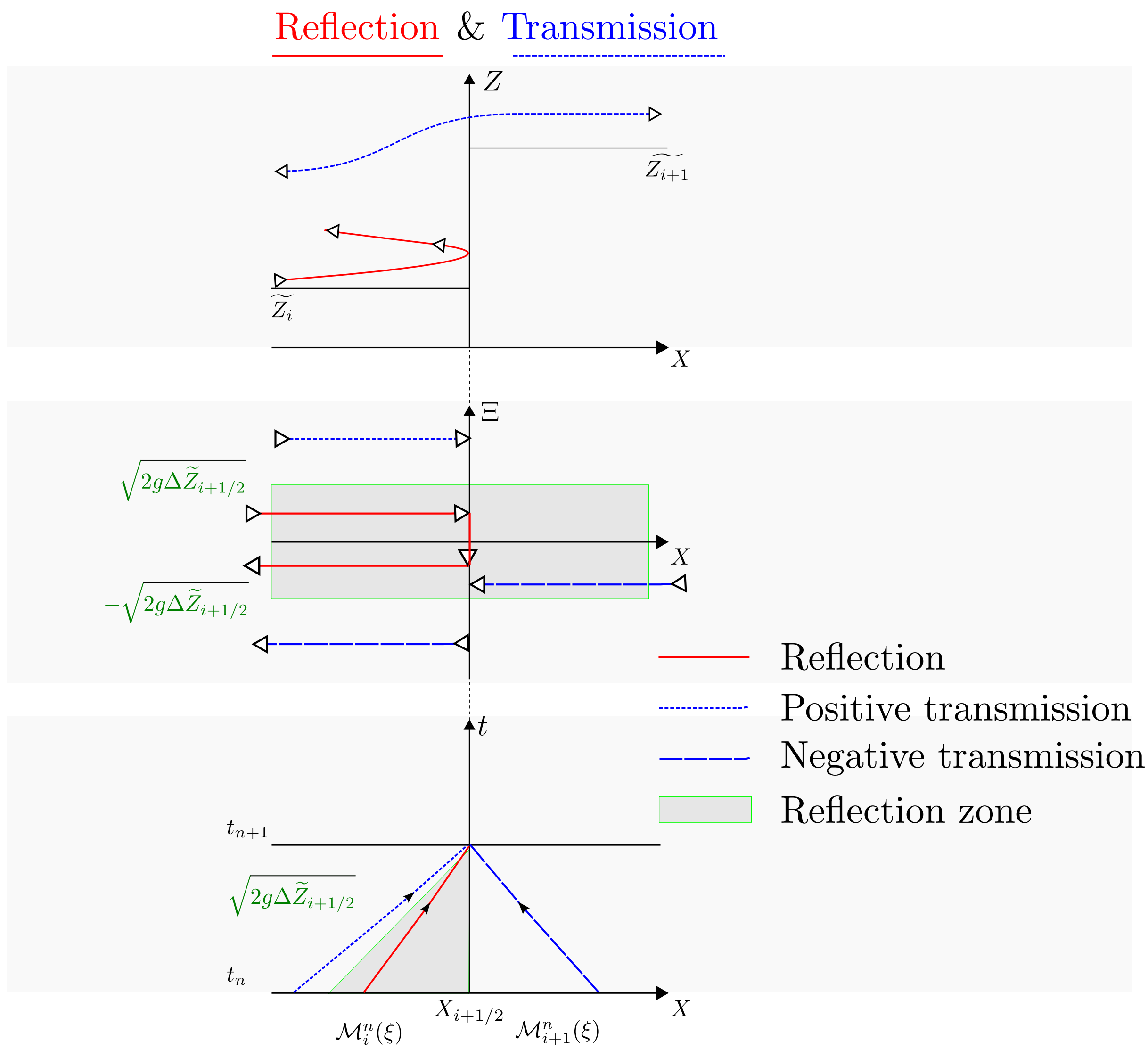



Figure 4: The potential barrier: transmission and reflection of particle
Top: the physical configuration
Middle: the characteristic solution in $(X, \Xi)$-plane
Bottom: the characteristic solution in $(X, t)$-plane

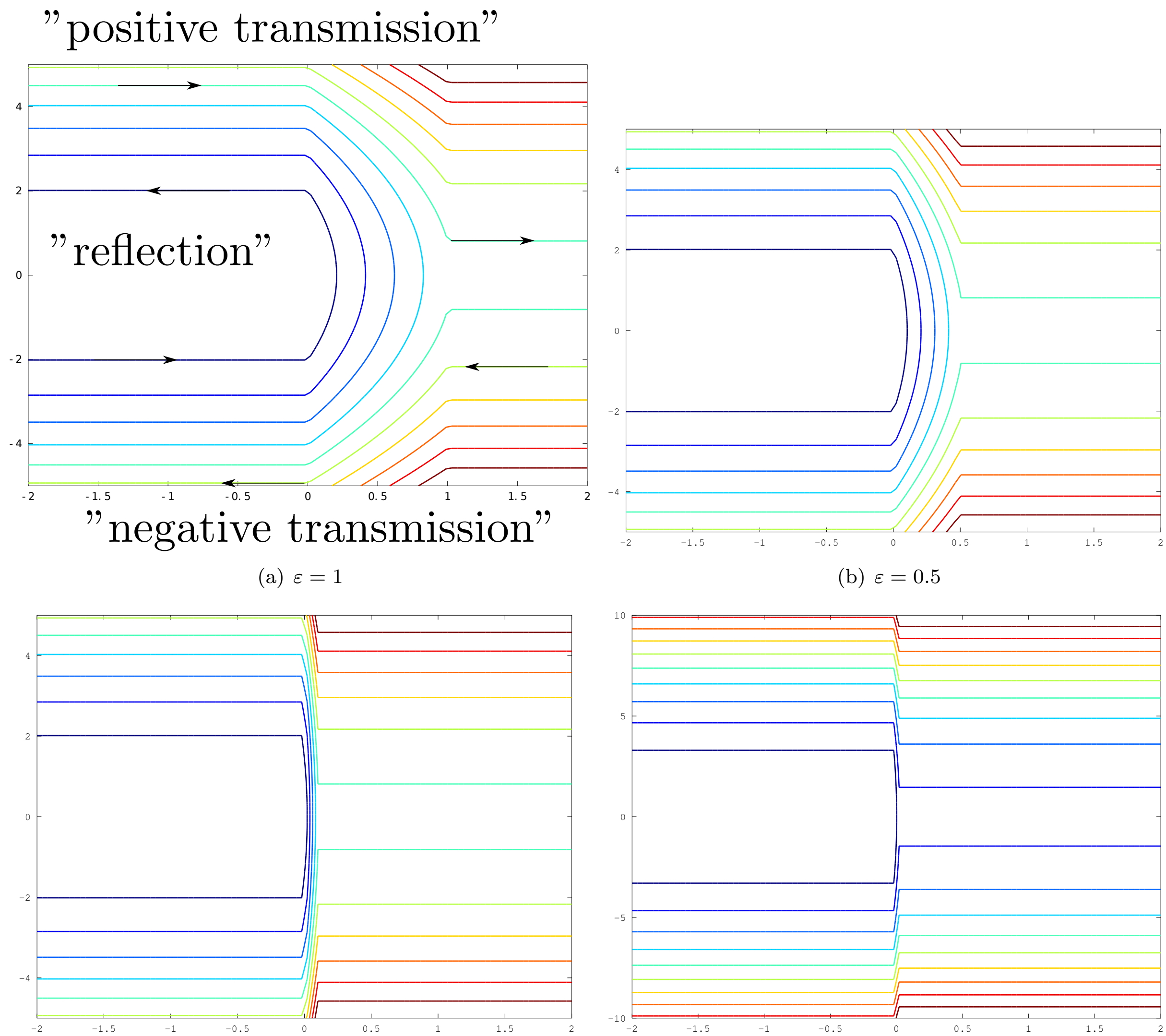
**Remark A.1.** Discontinuous trajectories can be also seen as the formal limit of the following problem as $\varepsilon$ vanishes

$$\begin{aligned}\frac{d\,\Xi}{ds} &= -g\partial_x Z_\varepsilon(X(s)) \\ \frac{d\,X}{ds} &= \Xi(s)\end{aligned}$$

where $Z_\varepsilon$ is a smooth function which satisfies $\lim_{\varepsilon\to 0} Z_\varepsilon = Z$.

On figure 5, we represent the contour plot of $\Xi^2 + 2gZ_\varepsilon(X)$ as $\varepsilon$ decreases where

$$Z_\varepsilon(x) = \begin{cases} 0 & \text{if} \quad x < 0 \quad , \\ \dfrac{x}{\varepsilon} & \text{if} \quad 0 < x < \varepsilon \quad , \\ 1 & \text{otherwise} \quad . \end{cases}$$




(a) $\varepsilon = 1$

(b) $\varepsilon = 0.5$

(c) $\varepsilon = 0.1$

(d) $\varepsilon = 0.001$

Figure 5: Contour plot of $\Xi^2 + 2gZ_\varepsilon(X)$

To compute the microscopic flux $\mathcal{M}^-_{i+1/2}$ (12) (the second one being computed in a same way, it will be not treated), we proceed as follows.

- Assume that $\xi_{n+1} > 0$. Then we deduce from Equation (27) and under well-suited CFL stability condition (see [19, Proof of Theorem 3]), that $X(t_n) \in (x_{i-1/2}, x_{i+1/2})$ and $\xi_n = \xi_{n+1}$. It means that the particle comes from the left hand side of the interface $x_{i+1/2}$. Thus the microscopic flux is given by

$$\mathcal{M}^-_{i+1/2}(\xi) = \mathcal{M}_i(\xi)\mathbb{1}_{\xi>0} \tag{28}$$

which corresponds to the so-called "positive transmission".

- Assume that $\xi_{n+1} < 0$ and $|\xi_n|^2 - 2g\Delta Z_{i+1/2} < 0$. Then, we deduce from Equation (27) that $X(t_n) \in (x_{i-1/2}, x_{i+1/2})$. As a consequence, in view of Equation (26), $\Delta Z_{i+1/2} \equiv 0$ and $\xi_n = -\xi_{n+1}$ with $\xi_n > 0$. From an energetic consideration, it means that the particle has not enough kinetic energy $\frac{\xi_n^2}{2}$ to pass the potential barrier $\Delta Z_{i+1/2} = Z_{i+1} - Z_i$ and thus it is reflected. Consequently, one has

$$\mathcal{M}^-_{i+1/2}(\xi) = \mathcal{M}_i(-\xi)\mathbb{1}_{\xi<0, |\xi|^2 - 2g\Delta Z_{i+1/2}<0}(\xi)\ . \tag{29}$$

- Finally, assume that $\xi_{n+1} < 0$ and $|\xi_{n+1}|^2 = |\xi_n|^2 - 2g\Delta Z_{i+1/2} > 0$. We deduce that $X(t_n) \in (x_{i+1/2}, x_{i+3/2})$ and $\xi_{n+1} = -\sqrt{|\xi_n|^2 - 2g\Delta Z_{i+1/2}}$. It means that the particle comes from the right hand side of the interface $x_{i+1/2}$ with enough kinetic energy to pass the potential barrier. In this case, the flux is

$$\mathcal{M}^-_{i+1/2}(\xi) = \mathcal{M}_{i+1}\left(-\sqrt{|\xi|2 - 2g\Delta Z_{i+1/2}}\right)\mathbb{1}_{\xi<0, |\xi|^2 - 2g\Delta Z_{i+1/2}>0}(\xi)\ . \tag{30}$$

This is the so-called "negative transmission".

Finally, gathering results (28)–(30), we obtain the microscopic fluxes (12).

**Remark A.2.** *Finally, let us outline that in the initial work by Perthame and Simeoni [19], the quantities $\Delta Z_{i+1/2}$ and $\Delta Z_{i-1/2}$ are not defined. These quantities are precisely defined as follows*

$$\Delta Z_{i+1/2} = Z_{i+1} - Z_i \quad \textit{and} \quad \Delta Z_{i-1/2} = Z_i - Z_{i-1}\ .$$

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